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NOTE ON STIRLING'S FORMULA

BY S. A. COREY

STIRLING's formula,* written in the notation ordinarily employed for Taylor's formula, is

$$\begin{aligned} f(a+x) &= f(a) + \frac{x}{2}\left[f'(a+x)+f'(a)\right] - \frac{B_1x^2}{2\,!}\left[f''(a+x)-f''(a)\right] \\ &\quad + \frac{B_2x^4}{4\,!}\left[f^{\text{iv}}(a+x)-f^{\text{iv}}(a)\right] - \cdots \\ &\quad + (-1)^n\frac{B_nx^{2n}}{(2n)\,!}\left[f^{[2n]}(a+x)-f^{[2n]}(a)\right] + \cdots. \qquad (1) \end{aligned}$$

Here B_1, B_2, $\cdots$ are Bernoulli's numbers, as follows: $B_1 = 1/6$, $B_2 = 1/30$, $B_3 = 1/42$, $B_4 = 1/30$, $B_5 = 5/66$, $B_6 = 691/2730$, $B_7 = 7/6$, etc.

If $$f\left(a+\frac{x}{m}\right),\ f\left[\left(a+\frac{2x}{m}\right)+\frac{x}{m}\right],\ f\left[\left(a+\frac{3x}{m}\right)+\frac{x}{m}\right],\ \cdots,\ f\left[\left(a+\frac{m-1}{m}x\right)+\frac{x}{m}\right]$$

be developed successively by Stirling's formula, simplifying each by the aid of the preceding, the following formula is readily obtained:

$$\begin{aligned} f(a+x) &= f(a) + \frac{x}{m\cdot 2}\Bigg\{\left[f'(a+x)+f'(a)\right] \\ &\quad + 2\left[f'\left(a+\frac{x}{m}\right)+f'\left(a+\frac{2x}{m}\right)+f'\left(a+\frac{3x}{m}\right)+\cdots+f'\left(a+\frac{m-1}{m}x\right)\right]\Bigg\} \\ &\quad - \frac{B_1x^2}{m^2\cdot 2\,!}\left[f''(a+x)-f''(a)\right] + \frac{B_2x^4}{m^4.4\,!}\left[f^{\text{iv}}(a+x)-f^{\text{iv}}(a)\right] - \cdots \\ &\quad + (-1)^n\frac{B_n\,x^{2n}}{m^{2n}\cdot(2n)\,!}\left[f^{[2n]}(a+x)-f^{[2n]}(a)\right] + \qquad (2) \end{aligned}$$

This formula (2) is admirably adapted for the numerical computation of functions for which the value of the first derivative for each of the successive and equidistant steps, a, $a + x/m$, $a + 2x/m$, $a + 3x/m$, $\cdots$, $a + x$, is known or may be readily found, or of functions whose successive derivatives increase rapidly in complexity with the order of the derivatives. The special value of the formula will be understood from the following considerations :—

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* See Bertrand's *Calcul différentiel*, p. 332.

1. No odd-numbered derivatives, except the first derivative, enter into the formula and their computation is therefore not required, except in so far as may be necessary to determine the form of the even-numbered derivatives of higher orders.

2. The values of the even-numbered derivatives need be found for only two values of the variable, viz., $x = 0$ and $x = x$, the values of such derivatives for the intermediate steps having been entirely eliminated from the formula.

3. The value of the general term may be made as small as desired by making m large enough, that is by finding the value of the first derivative for a sufficiently large number of intermediate points, corresponding to equidistant values of x.

As (2) is but the sum of a finite number of series developed by Stirling's formula, it follows that (2) is convergent if each of the component series is convergent, and, consequently, that the same test which determines the convergence of each of the component series determines the convergence of (2).

The following examples will serve to illustrate the usefulness of formula (2) for numerical computation. In each example the value of m and the highest value of n used are given, and the results obtained are correct to the last place of decimals.

$$(1) \qquad \int_0^1 \frac{dx}{1 + x} = 0.6931471805 \qquad (m = 10,\ n = 3),$$

$$(2) \qquad \int_0^{3\pi} \frac{\sin x}{x}\, dx = 1.67476 \qquad (m = 10,\ n = 2),$$

$$(3) \qquad \int_0^{\frac{\pi}{4}} \tan x\, dx = 0.346573 \qquad (m = 10,\ n = 1),$$

$$(4) \qquad \int_0^{\frac{\pi}{4}} \left(\frac{2 - x^2}{2 - 2x^2}\right)^{\frac{1}{2}} dx = 0.84829 \qquad (m = 10,\ n = 1).$$

Hiteman, Iowa, October 1903.